A note on a complex Hilbert metric with application to domain of analyticity for entropy rate of hidden Markov processes

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Abstract

In this note, we show that small complex perturbations of positive matrices are contractions, with respect to a complex version of the Hilbert metric, on the standard complex simplex. We show that this metric can be used to obtain estimates of the domain of analyticity of entropy rate for a hidden Markov process when the underlying Markov chain has strictly positive transition probabilities.

The purpose of this note is twofold. First, in Section 1, we introduce a complex version of the Hilbert metric on the standard real simplex. This metric is defined on a complex neighbourhood of the interior of the standard real simplex, within the standard complex simplex. We show that if the neighbourhood is sufficiently small, then for any sufficiently small complex perturbation of a strictly positive square matrix acts as a contraction, with respect to this metric. While this paper was nearing completion, we were informed of a different complex Hilbert metric, which was recently introduced. We briefly discuss the relation between this metric [2] and our metric in Remark 1.6.

Secondly, we show how one can use a complex Hilbert metric to obtain lower estimates of the domain of analyticity of entropy rate for a hidden Markov process when the underlying Markov chain has strictly positive transition probabilities. The domain of analyticity is important because it specifies an explicit region where a Taylor series converges to the entropy rate and also gives an explicit estimate on the rate of convergence of the Taylor approximation.

In principle, an estimate on the domain can be obtained by examining the proof of analyticity in [5]. That proof was based on a contraction mapping argument, using the fact that the real Euclidean metric is equivalent to the real Hilbert metric. In Section 2.1, we revisit certain aspects of the proof and outline how to modify the proof using a complex Hilbert metric; this yields a more direct estimate. In Section 2.2, we illustrate this with a small example, using our Hilbert metric.

We remark that the entropy rate of a hidden Markov process can be interpreted as a top Lyapunov exponent for a random matrix product [4]. In principle, a complex Hilbert metric can be used, more generally, to estimate the domain of analyticity of the top Lyapunov exponent for certain random matrix products; see [8], [9].

1 Complex Hilbert Metric

We begin with a review of the real Hilbert metric. Let B be a positive integer, and let W be the standard simplex in B-dimensional real Euclidean space:

$$W = \{w = (w_1, w_2, \cdots, w_B) \in \mathbb{R}^B : w_i \geq 0, \sum_i w_i = 1\},$$

and let W° denote its interior, consisting of the vectors with positive coordinates. For any two vectors $v, w \in W^\circ$, the Hilbert metric [12] is defined as

$$d_H(w, v) = \max_{i,j} \log \left(\frac{w_i/w_j}{v_i/v_j} \right). \tag{1}$$

For a $B \times B$ strictly positive matrix $T = (t_{ij})$, the mapping f_T induced by T on W is defined by $f_T(w) = wT/(wT\mathbf{1})$, where $\mathbf{1}$ is the all-ones vector. It is well known that f_T is a contraction mapping under the Hilbert metric [12]. The contraction coefficient of T, which is also called the Birkhoff coefficient, is given by:

$$\tau(T) = \sup_{v \neq w} \frac{d_H(vT, wT)}{d_H(v, w)} = \frac{1 - \sqrt{\phi(T)}}{1 + \sqrt{\phi(T)}}, \tag{2}$$

where $\phi(T) = \min_{i,j,k,l} \frac{t_{ik} t_{jl}}{t_{jk} t_{il}}$. This result extends to the case where T has all columns strictly positive or all zero and at least one strictly positive column (then, in the definition of $\phi(T)$, consider only k, l corresponding to strictly positive columns).

Let $W_\mathbb{C}$ denote the complex version of W, i.e., $W_\mathbb{C}$ denotes the complex simplex comprising the vectors

$$\{w = (w_1, w_2, \cdots, w_B) \in \mathbb{C}^B : \sum_i w_i = 1\}.$$

Let $W_\mathbb{C}^+ = \{v \in W_\mathbb{C} : \mathcal{R}(v_i) > 0\}$. For $v, w \in W_\mathbb{C}^+$, let

$$d_H(v, w) = \max_{i,j} \left| \log \left(\frac{w_i/w_j}{v_i/v_j} \right) \right|, \tag{3}$$

where log is taken as the principal branch of the complex $\log(\cdot)$ function (i.e., the branch whose branch cut is the negative real axis). Since the principal branch of log is additive on the right-half plane, d_H is a metric on $W_\mathbb{C}^+$, which we call a *complex Hilbert metric.*

We begin with the following very simple lemma.

Lemma 1.1. *Let $n \geq 2$. For any fixed $z_1, z_2, \cdots, z_n, z \in \mathbb{C}$ and fixed $t > 0$, we have*

$$\sup_{t_1,\cdots,t_n \geq 0,\ t_1+t_2+\cdots+t_n=t} |t_1 z_1 + t_2 z_2 + \cdots + t_n z_n + z| = \max_{i=1,\cdots,n} |t z_i + z| .$$

Proof. The convex hull of $z_1, z_2, \cdots, z_n$ is a solid polygon, taking the form

$$\{(t_1/t)z_1 + (t_2/t)z_2 + \cdots + (t_n/t)z_n : t_1, t_2, \cdots, t_n \geq 0, t_1 + t_2 + \cdots + t_n = t\}.$$

By convexity, the distance from any point in this solid polygon to the point $(-1/t)z$ will achieve the maximum at one of the extreme points, namely

$$\sup_{t_1,\cdots,t_n \geq 0, t_1+t_2+\cdots+t_n=t} |(t_1/t)z_1 + (t_2/t)z_2 + \cdots + (t_n/t)z_n - (-1/t)z| = \max_{i=1,\cdots,n} |z_i - (-1/t)z| .$$

The lemma then immediately follows.

□

The following lemma is implied by the proof of Lemma 2.1 of [10]; we give a proof for completeness.

Lemma 1.2. *For fixed $a_1, a_2, \cdots, a_B > 0 \in \mathbb{R}$ and fixed $x_1, x_2 \cdots, x_B > 0 \in \mathbb{R}$, define:*

$$D_n = \frac{a_n x_n}{\sum_{m=1}^B a_m x_m} - \frac{x_n}{\sum_{m=1}^B x_m}.$$

Let $\mathcal{T}_0 = \{n : D_n \geq 0\}$ and $\mathcal{T}_1 = \{n : D_n < 0\}$. Then we have

$$\sum_{n \in \mathcal{T}_0} D_n = \sum_{n \in \mathcal{T}_1} |D_n| \leq \frac{1 - \sqrt{a/A}}{1 + \sqrt{a/A}},$$

where $a = \min\{a_1, a_2, \cdots, a_B\}$ and $A = \max\{a_1, a_2, \cdots, a_B\}$.

Proof. It immediately follows from $\sum_{n=1}^B D_n = 0$ and the definitions of $\mathcal{T}_0$ and $\mathcal{T}_1$ that

$$\sum_{n \in \mathcal{T}_0} D_n = \sum_{n \in \mathcal{T}_1} |D_n|.$$

Now

$$\begin{aligned}\sum_{n \in \mathcal{T}_0} D_n &= \sum_{n \in \mathcal{T}_0} \left(\frac{a_n x_n}{\sum_{m \in \mathcal{T}_0} a_m x_m + \sum_{m \in \mathcal{T}_1} a_m x_m} - \frac{x_n}{\sum_{m \in \mathcal{T}_0} x_m + \sum_{m \in \mathcal{T}_1} x_m} \right) \\ &\leq \sum_{n \in \mathcal{T}_0} \left(\frac{A x_n}{A \sum_{m \in \mathcal{T}_0} x_m + a \sum_{m \in \mathcal{T}_1} x_m} - \frac{x_n}{\sum_{m \in \mathcal{T}_0} x_m + \sum_{m \in \mathcal{T}_1} x_m} \right)\end{aligned}$$

Let

$$z = \frac{\sum_{m \in \mathcal{T}_1} x_m}{\sum_{m \in \mathcal{T}_0} x_m},$$

we then have

$$\sum_{n \in \mathcal{T}_0} D_n \leq \frac{1}{1 + (a/A)z} - \frac{1}{1+z} = f(z).$$

Simple calculus shows that $f(z)$ will be bounded above by $\frac{1-\sqrt{a/A}}{1+\sqrt{a/A}}$ on $[0, \infty)$. This establishes the lemma. □

Let $W^\circ_{\mathbb{C}}(\delta)$ denote the "relative" δ-neighborhood of W° in $W_{\mathbb{C}}$, i.e.,

$$W^\circ_{\mathbb{C}}(\delta) = \{v = (v_1, v_2, \cdots, v_B) \in W_{\mathbb{C}} : \exists u \in W^\circ, |v_i - u_i| \leq \delta |u_i|, i = 1, 2, \cdots, B\}.$$

Note that when $\delta < 1$, $W^\circ_{\mathbb{C}}(\delta) \subset W^+_{\mathbb{C}}$ and so the complex Hilbert metric is defined on $W^\circ_{\mathbb{C}}(\delta)$.

We consider complex matrices $\hat{T} = (\hat{t}_{ij})$ which are perturbations of a strictly positive matrix $T = (t_{ij})$. For such a matrix T and $r > 0$, let $B_T(r)$ denote the set of all complex matrices $\hat{T}$ such that for all i, j,

$$|t_{ij} - \hat{t}_{ij}| \leq r.$$

With the aid of the above lemmas, we shall prove:

Theorem 1.3. *Let T be a strictly positive matrix. There exist $r, \delta > 0$ such that whenever $\hat{T} \in B_T(r)$, $f_{\hat{T}}$ is a contraction mapping on $W^\circ_{\mathbb{C}}(\delta)$ under the complex Hilbert metric.*

Proof. For $\hat{x}, \hat{y} \in W^+_{\mathbb{C}}$, $\hat{x} \neq \hat{y}$, and i, j, let

$$L_{ij} = \frac{\log(\sum_m \hat{x}_m \hat{T}_{mi} / \sum_m \hat{x}_m \hat{T}_{mj}) - \log(\sum_m \hat{y}_m \hat{T}_{mi} / \hat{y}_m \hat{T}_{mj})}{\max_{k,l} |\log(\hat{x}_k/\hat{y}_k) - \log(\hat{x}_l/\hat{y}_l)|}.$$

Note that

$$\frac{d_H(\hat{x}\hat{T}, \hat{y}\hat{T})}{d_H(\hat{x}, \hat{y})} = \max_{i,j} |L_{ij}|.$$

It suffices to prove that there exists $0 < \rho < 1$ such that for sufficiently small $r, \delta > 0$, $\hat{x}, \hat{y} \in W^\circ_{\mathbb{C}}(\delta)$, $\hat{x} \neq \hat{y}$, $\hat{T} \in B_T(r)$, and any i, j,

$$|L_{ij}| < \rho.$$

For each m, let $\hat{c}_m = \log \hat{x}_m / \hat{y}_m$; then $\hat{x}_m = \hat{y}_m e^{\hat{c}_m}$. Choose $p \neq q$ such that

$$|\hat{c}_p - \hat{c}_q| = \max_{k,l} |\hat{c}_k - \hat{c}_l|.$$

Hence:

$$L_{ij} = \frac{\log(\sum_m \hat{y}_m e^{\hat{c}_m - \hat{c}_q} \hat{T}_{mi} / \sum_m \hat{y}_m e^{\hat{c}_m - \hat{c}_q} \hat{T}_{mj}) - \log(\sum_m \hat{y}_m \hat{T}_{mi} / \hat{y}_m \hat{T}_{mj})}{|\hat{c}_p - \hat{c}_q|}.$$

Define

$$F(t) = \log(\sum_m \hat{y}_m e^{(\hat{c}_m - \hat{c}_q)t} \hat{T}_{mi} / \sum_m \hat{y}_m e^{(\hat{c}_m - \hat{c}_q)t} \hat{T}_{mj}).$$

Since

$$|F(1) - F(0)| = \left| \int_0^1 F'(t) dt \right| \leq \max_{\xi \in [0,1]} |F'(\xi)|,$$

we have

$$|L_{ij}| = \frac{|F(1) - F(0)|}{|\hat{c}_p - \hat{c}_q|} \leq \frac{\max_{\xi \in [0,1]} |F'(\xi)|}{|\hat{c}_p - \hat{c}_q|}. \tag{4}$$

Note that $F'(\xi)$ takes the following form:

$$F'(\xi) = \frac{\sum_m (\hat{c}_m - \hat{c}_q)\hat{y}_m e^{(\hat{c}_m-\hat{c}_q)\xi}\hat{T}_{mi}}{\sum_m \hat{y}_m e^{(\hat{c}_m-\hat{c}_q)\xi}\hat{T}_{mi}} - \frac{\sum_m (\hat{c}_m - \hat{c}_q)\hat{y}_m e^{(\hat{c}_m-\hat{c}_q)\xi}\hat{T}_{mj}}{\sum_m \hat{y}_m e^{(\hat{c}_m-\hat{c}_q)\xi}\hat{T}_{mj}}$$

Now for all m let $\hat{a}_m = \hat{T}_{mi}/\hat{T}_{mj}$. Then

$$\frac{F'(\xi)}{|\hat{c}_p - \hat{c}_q|} = \sum_n \frac{\hat{c}_n - \hat{c}_q}{|\hat{c}_p - \hat{c}_q|}\left(\frac{\hat{y}_n e^{(\hat{c}_n-\hat{c}_q)\xi}\hat{a}_n\hat{T}_{nj}}{\sum_m \hat{y}_m e^{(\hat{c}_m-\hat{c}_q)\xi}\hat{a}_m\hat{T}_{mj}} - \frac{\hat{y}_n e^{(\hat{c}_n-\hat{c}_q)\xi}\hat{T}_{nj}}{\sum_m \hat{y}_m e^{(\hat{c}_m-\hat{c}_q)\xi}\hat{T}_{mj}}\right) = \sum_n \frac{\hat{c}_n - \hat{c}_q}{|\hat{c}_p - \hat{c}_q|}\hat{D}_n. \tag{5}$$

where $\hat{D}_n$ denotes the quantity in parentheses in the middle expression above.

Let $x, y \in W^\circ$ such that for all k, $|\hat{x}_k - x_k| \le \delta|x_k|$ and $|\hat{y}_k - y_k| \le \delta|y_k|$. Let $a_m = T_{mi}/T_{mj}$, $c_m = \log x_m/y_m$, and let D_n denote the unperturbed version of $\hat{D}_n$:

$$D_n = \frac{y_n e^{(c_n-c_q)\xi}a_n T_{nj}}{\sum_m y_m e^{(c_m-c_q)\xi}a_m T_{mj}} - \frac{y_n e^{(c_n-c_q)\xi}T_{nj}}{\sum_m y_m e^{(c_m-c_q)\xi}T_{mj}}. \tag{6}$$

By Lemma 1.2, we have

$$\sum_{n\in\mathcal{T}_0} D_n = \sum_{n\in\mathcal{T}_1} |D_n| \le \max_{k,l} \frac{1-\sqrt{a_k/a_l}}{1+\sqrt{a_k/a_l}} \le \tau(T), \tag{7}$$

where $\mathcal{T}_0 = \{n : D_n \ge 0\}$ and $\mathcal{T}_1 = \{n : D_n < 0\}$.

Now, for some universal constant K_0,

$$\left|\sum_n \frac{\hat{c}_n - \hat{c}_q}{|\hat{c}_p - \hat{c}_q|}\hat{D}_n - \sum_n \frac{\hat{c}_n - \hat{c}_q}{|\hat{c}_p - \hat{c}_q|}D_n\right| < K_0(\delta + r). \tag{8}$$

Applying Lemma 1.1 twice, we conclude that there exist $n_0 \in \mathcal{T}_0, n_1 \in \mathcal{T}_1$ such that

$$\left|\sum_n \frac{\hat{c}_n - \hat{c}_q}{|\hat{c}_p - \hat{c}_q|}D_n\right|$$

$$\le \left|\frac{\hat{c}_{n_0} - \hat{c}_q}{|\hat{c}_p - \hat{c}_q|}\sum_{n\in\mathcal{T}_0} D_n + \sum_{n\in\mathcal{T}_1}\frac{\hat{c}_n - \hat{c}_q}{|\hat{c}_p - \hat{c}_q|}D_n\right| \le \left|\frac{\hat{c}_{n_0} - \hat{c}_q}{|\hat{c}_p - \hat{c}_q|}\sum_{n\in\mathcal{T}_0} D_n - \frac{\hat{c}_{n_1} - \hat{c}_q}{|\hat{c}_p - \hat{c}_q|}\sum_{n\in\mathcal{T}_1}|D_n|\right|.$$

Then together with (4), (5), (8), (7), and the fact that $|\hat{c}_{n_1} - \hat{c}_{n_0}| \le |\hat{c}_p - \hat{c}_q|$, we obtain that for sufficiently small $r, \delta > 0$, $|L_{ij}|$ is upper bounded by some $\rho < 1$, as desired. □

Remark 1.4. One can further choose $r, \delta > 0$ such that when $\hat{T} \in B_T(r)$, $f_{\hat{T}}(W^\circ_{\mathbb{C}}(\delta)) \subset W^\circ_{\mathbb{C}}(\delta)$. Consider a compact subset $N \subset W^\circ$ such that $f_T(W) \subset N$. Let $N(R)$ denote the Euclidean R-neighborhood of N in $W_{\mathbb{C}}$. The proof of Theorem 1.3 implies that when $T > 0$ or ($T \ge 0$ and $\sup_{x,y\in N, 0\le\xi\le 1}\sum_{n\in\tau_0} D_n < 1$ (here D_n is defined in (6))), there exist $r, R > 0$ such that when $\hat{T} \in B_T(r)$, $f_{\hat{T}}$ is a contraction mapping on $N(R)$ under the complex Hilbert metric.

Example 1.5. Consider a 2×2 strictly positive matrix

$$T = \left[\begin{array}{cc} a & c \\ b & d \end{array} \right].$$

If we parameterize the interior of the simplex W° by $(0, \infty)$: $w = (x, y) \mapsto x/y$, then letting $z = x/y$, we have: $f_T(z) = \frac{az+b}{cz+d}$; the domain of this mapping naturally extends from $(0, \infty)$ to the open right half complex plane H, and the complex Hilbert metric becomes simply $d_H(z_1, z_2) = |\log(z_1/z_2)|$.

One can show that f_T is a contraction on all of H with contraction coefficient:

$$\tau(T) = \frac{1 - \frac{bc}{ad}}{1 + \frac{bc}{ad}}.$$

(assuming $\det(T) \geq 0$; otherwise, the last expression is replaced by $\frac{1-\frac{ad}{bc}}{1+\frac{ad}{bc}}$). To see this, for any $z, w \in H$, consider

$$L = \left| \frac{\log(f_T(z)) - \log(f_T(w))}{\log(z) - \log(w)} \right|.$$

With change of variables $u = \log(z), v = \log(w)$, we have

$$L = \left| \frac{\log(f_T(e^u)) - \log(f_T(e^v))}{u - v} \right| = \left| \int_0^1 e^{v+t(u-v)} \frac{f_T'(e^{v+t(u-v)})}{f_T(e^{v+t(u-v)})} dt \right|,$$

which implies that

$$L \leq \sup_{z \in H} \left| \frac{z f_T'(z)}{f_T(z)} \right|.$$

A simple computation shows that

$$\frac{z f_T'(z)}{f_T(z)} = \frac{ad - bc}{acz + (ad + bc) + bd/z}. \tag{9}$$

To see that the supremum is $\frac{1-\frac{bc}{ad}}{1+\frac{bc}{ad}}$, first note that since $ad - bc \geq 0$ and $a, b, c, d > 0$, the absolute value of the quantity on the right-hand side of (9) is maximized by minimizing $|acz + bd/z|$; since the only solutions to $acz + bd/z = 0$ are $z = \pm i\sqrt{bd/ac}$, one sees that the supremum is obtained by substituting $z = \pm i\sqrt{bd/ac}$ into (9), and this shows that the supremum is indeed $\frac{1-\frac{bc}{ad}}{1+\frac{bc}{ad}}$.

Note that this contraction coefficient on H is strictly larger (i.e., worse) than the contraction coefficient on $[0, \infty)$: $\frac{1-\sqrt{\frac{bc}{ad}}}{1+\sqrt{\frac{bc}{ad}}}$.

When

$$\hat{T} = \left[\begin{array}{cc} \hat{a} & \hat{c} \\ \hat{b} & \hat{d} \end{array} \right].$$

is a sufficiently small complex perturbation of T, then $f_{\hat{T}}(H) \subseteq H$ and one obtains

$$\tau(\hat{T}) = \sup_{z \in H} \left| \frac{z f_{\hat{T}}'(z)}{f_{\hat{T}}(z)} \right| = \sup_{z \in H} \left| \frac{\hat{a}\hat{d} - \hat{b}\hat{c}}{\hat{a}\hat{c}z + (\hat{a}\hat{d} + \hat{b}\hat{c}) + \hat{b}\hat{d}/z} \right|$$

which will approximate $\frac{1-\frac{bc}{ad}}{1+\frac{bc}{ad}}$, and so $f_{\hat{T}}$ will still be a contraction on H.

Remark 1.6. While this paper was nearing completion, we were informed that alternative complex Hilbert metrics, based on the Poincare metric in the right-half complex plane, were recently introduced in Rugh [11] and Dubois [2]. Contractiveness with respect to these metrics is proven in great generality and yields far-reaching consequences for complex Perron-Frobenius theory. The proofs of contractiveness in these papers seem rather different from the calculus approach in our paper.

The complex Hilbert metric, which we call d_P, used in [2] (see equation (3.23)) is explicit and natural, but slightly more complicated than our complex Hilbert metric; for $v, w \in W_{\mathbb{C}}^+$,

$$d_P(w,v) = \log \frac{\max_{i,j}(|\overline{w_i}v_j + \overline{w_j}v_i| + |w_iv_j - w_jv_i|)(2\mathcal{R}(\overline{w_i}w_j))^{-1}}{\min_{i,j}(|\overline{w_i}v_j + \overline{w_j}v_i| - |w_iv_j - w_jv_i|)(2\mathcal{R}(\overline{w_i}w_j))^{-1}}; \tag{10}$$

here $\overline{z}$ denotes complex conjugate, $\mathcal{R}(z)$ denotes real part, and log is the ordinary real logarithm. In the 2-dimensional case, it can be verified that, if one transforms $w = (w_1, w_2)$ and $v = (v_1, v_2)$ to $z_1 = w_2/w_1$ and $z_2 = w_2/w_1$, then d_P reduces to the Poincare metric on H:

$$d_P(z_1, z_2) = \log \frac{|z_1 + \bar{z}_2| + |z_1 - z_2|}{|z_1 + \bar{z}_2| - |z_1 - z_2|}.$$

Using the infinitesimal form for the Poincare metric (as a Riemannian metric on H), one checks that, in the 2×2 case, the Lipschitz constant for a complex matrix $\hat{T}$ such that $f_{\hat{T}}(H) \subseteq H$ is:

$$\sup_{z \in H} \left| \frac{\mathcal{R}(z) f'_{\hat{T}}(z)}{\mathcal{R}(f_{\hat{T}}(z))} \right| \tag{11}$$

in contrast to

$$\sup_{z \in H} \left| \frac{z f'_{\hat{T}}(z)}{f_{\hat{T}}(z)} \right| \tag{12}$$

for our complex Hilbert metric (as in Example 1.5 above).

While we have not analyzed in detail the differences between these metrics, there are a few things that can be said in the 2×2 case:

- $f_{\hat{T}}$ is a contraction with respect to d_P on H whenever it maps H into its interior; this follows from standard complex analysis (section IX.3 of of [3]), and Dubois [2] proves an analog of this for the metric d_P above (10) in higher dimensions. However, this does not hold for d_H.

- When $\hat{T} = T$ is strictly positive, then the contraction coefficient, with respect to d_P, is always at least as good (i.e., at most) the contraction coefficient with respect to d_H. This can be seen as follows:

  First recall that any fractional linear transformation T can be expressed as the composition of transitions, dilations and inversions. In the case where T is strictly positive, the translations are by positive real numbers and the dilations are by real numbers;

see page 65 of [3]. Using the infinitesimal forms (11, 12), our assertion would follow from:

$$\left|\frac{\mathcal{R}(z)}{z}\right| \leq \left|\frac{\mathcal{R}(f_T(z))}{f_T(z)}\right|, \quad \text{for all } z \in H. \tag{13}$$

This is true indeed: it is easy to see that in fact we get equality in (13) for inversions and dilations by real numbers, and we get strict inequality in (13) for translations by positive real numbers.

- When $\hat{T}$ is a complex perturbation of a strictly positive T, then (13) (with T replaced by $\hat{T}$) need not hold; in fact, for perturbations $\hat{T}$ of T on the order of 1% and $z = x + yi \in H$, with $|y|/x$ on the order of 1%, the contraction coefficient with respect to d_H may be slightly smaller than that with respect to d_P. The reason is that in this case, the dilations may be complex (non-real) and for such a dilation the inequality (13) may be reversed. Examples of this can be randomly generated in Matlab. For example, if

$$\hat{T} = \begin{bmatrix} 0.012890500224 + 0.000128905002i & 0.310402226067 + 0.003104022260i \\ 0.779079247486 - 0.007790792474i & 0.307296084921 - 0.003072960849i \end{bmatrix}$$

  and $z = 0.926678310631 - 0.009266783106i$, then the contraction coefficent of d_H is approximately 0.664396 and that of d_P is approximately 0.664599. For larger perturbations, the differences in contraction coefficient can be greater. The relative strength of contraction of d_H, d_P seems to be heavily dependent on specific choices of $\hat{T}$ and z.

- For any point z, other than 0, of the imaginary axis, the metric d_H can be extended to a neighbourhood, with respect to which any sufficiently small complex perturbation $\hat{T}$ of a strictly positive matrix acts as a contraction; on the other hand, there is no way to do this with d_P since it blows up as one approaches the imaginary axis.

- Also, on a small punctured neighbourhood of 0, we replace d_H by the metric $d(z_1, z_2) = |\log(z_1) - \log(z_2)|$, then small complex perturbation $\hat{T}$ of a strictly positive matrix still acts as a contraction.

In the next section, we use d_H for estimates on the domain of analyticity of entropy rate of a hidden Markov process. Alternatively, d_P could be used, however it appears to be computationally easier to use d_H for the estimation.

2 Domain of Analyticity of Entropy Rate of Hidden Markov Processes

2.1 Background

For $m, n \in \mathbb{Z}$ with $m \leq n$, we denote a sequence of symbols $y_m, y_{m+1}, \ldots, y_n$ by y_m^n. Consider a stationary stochastic process Y with a finite set of states $\mathcal{I} = \{1, 2, \cdots, B\}$ and distribution

$p(y_m^n)$. Denote the conditional distributions by $p(y_{n+1}|y_m^n)$. The entropy rate of Y is defined as

$$H(Y) = \lim_{n\to\infty} -E_p(\log(p(y_0|y_{-n}^{-1}))),$$

where E_p denotes expectation with respect to the distribution p.

Let Y be a stationary first order Markov chain with

$$\Delta(i,j) = p(y_1 = j | y_0 = i).$$

In this section, we only consider the case when Δ is strictly positive.

A *hidden Markov process (HMP)* Z is a process of the form $Z = \Phi(Y)$, where Φ is a function defined on $\mathcal{I} = \{1, 2, \cdots, B\}$ with values in $\mathcal{J} = \{1, 2, \cdots, A\}$.

Recall that W is the B-dimensional real simplex and $W_{\mathbb{C}}$ is the complex version of W. For $a \in \mathcal{J}$, let $\mathcal{I}(a)$ denote the set of all indexes $i \in \mathcal{I}$ with $\Phi(i) = a$. Let

$$W_a = \{w \in W : w_i = 0 \text{ whenever } i \notin \mathcal{I}(a)\}$$

and

$$W_{a,\mathbb{C}} = \{w \in W_{\mathbb{C}} : w_i = 0 \text{ whenever } i \notin \mathcal{I}(a)\}.$$

Let Δ_a denote the $B \times B$ matrix such that $\Delta_a(i,j) = \Delta(i,j)$ for $j \in \mathcal{I}(a)$, and $\Delta_a(i,j) = 0$ for $j \notin \mathcal{I}(a)$ (i.e, Δ_a is formed from Δ by "zeroing out" the columns corresponding to indices that are not in $\mathcal{I}(a)$. the For $a \in \mathcal{J}$, define the scalar-valued and vector-valued functions r_a and f_a on W by

$$r_a(w) = w\Delta_a \mathbf{1},$$

and

$$f_a(w) = w\Delta_a / r_a(w).$$

Note that f_a defines the action of the matrix Δ_a on the simplex W. For any fixed n and z_{-n}^0 and for $i = -n, -n+1, \cdots$, define

$$x_i = x_i(z_{-n}^i) = p(y_i = \cdot \, | z_i, z_{i-1}, \cdots, z_{-n}), \tag{14}$$

(here $\cdot$ represent the states of the Markov chain Y); then from Blackwell [1], we have that $\{x_i\}$ satisfies the random dynamical iteration

$$x_{i+1} = f_{z_{i+1}}(x_i), \tag{15}$$

starting with

$$x_{-n-1} = p(y_{-n-1} = \cdot \,). \tag{16}$$

where $p(y_{-n-1} = \cdot\,)$ is the stationary distribution for the underlying Markov chain. One checks that $p(z_{i+1}|z_{-n}^i)$ can be recovered from this dynamical system; more specifically, we have

$$p(z_{i+1}|z_{-n}^i) = r_{z_{i+1}}(x_i).$$

If the entries of $\Delta = \Delta^{\vec{\varepsilon}}$ are analytically parameterized by a real variable vector $\vec{\varepsilon} \in \mathbb{R}^k$ (k is a positive integer), then we obtain a family $Z = Z^{\vec{\varepsilon}}$ and corresponding $\Delta_a = \Delta_a^{\vec{\varepsilon}}$, $f_a = f_a^{\vec{\varepsilon}}$, etc.

The following result was proven in [5].

Theorem 2.1. *Suppose that the entries of $\Delta = \Delta^{\vec{\varepsilon}}$ are analytically parameterized by a real variable vector $\vec{\varepsilon}$. If at $\vec{\varepsilon} = \vec{\varepsilon}_0$, Δ is strictly positive, then $H(Z) = H(Z^{\vec{\varepsilon}})$ is a real analytic function of $\vec{\varepsilon}$ at $\vec{\varepsilon}_0$.*

In [5] this result is stated in greater generality, allowing some entries of Δ to be zero. The proof is based on an analysis of the action of perturbations of f_a on neighbourhoods of $\hat{W}_b \stackrel{\triangle}{=} f_b(W)$, with respect to the Euclidean metric. The proof assumes that each f_a is a contraction on each $\hat{W}_b$. While this need not hold, one can arrange for this to be true by replacing the original system with a higher power system: namely, one replaces the original alphabet $\mathcal{J}$ with $\mathcal{J}^n$ for some n and replaces the mappings $\{f_a : a \in \mathcal{J}\}$ with $\{f_{a_0} \circ f_{a_1} \circ \cdots \circ f_{a_{n-1}} : a_0 a_1 \ldots a_{n-1} \in \mathcal{J}^n\}$. The existence of such an n follows from a) the equivalence of the (real) Hilbert metric and the Euclidean metric on each $\hat{W}_b$ (Proposition 2.1 of [5]) and b) the contractiveness of each f_a with respect to the (real) Hilbert metric. However, in the course of this replacement, one easily loses track of the domain of analyticity.

When at $\vec{\varepsilon} = \vec{\varepsilon}_0$, Δ is strictly positive, an alternative is to directly use a complex Hilbert metric, as follows. For each $a \in \mathcal{J}$, we can define a complex Hilbert metric $d_{a,H}$ on $W^{\circ}_{a,\mathbb{C}}$ as follows: for $w, v \in W^{\circ}_{a,\mathbb{C}}$:

$$d_{a,H}(w, v) = d_H(w_{\mathcal{I}(a)}, v_{\mathcal{I}(a)}) = \max_{i,j \in \mathcal{I}(a)} \left| \log \left(\frac{w_i/w_j}{v_i/v_j} \right) \right|. \tag{17}$$

Theorem 1.3 implies that for each $a, b \in \mathcal{J}$, sufficiently small perturbations of f_a are contractions on sufficiently small complex neighborhoods of $\hat{W}_b$ in $W_{b,\mathbb{C}}$; see Remark 1.4 (note that while Δ_a is not strictly positive, f_a maps into W_a and so as a mapping from W_b to W_a it can be regarded as the induced mapping of a strictly positive matrix). For complex $\vec{\varepsilon}$ close to $\vec{\varepsilon}_0$, $f_a = f_a^{\vec{\varepsilon}}$ is sufficiently close to $f_a^{\vec{\varepsilon}_0}$ to guarantee that $f_a^{\vec{\varepsilon}}$ is a contraction.

Let $\Omega_{a,H}(R)$ denote the neighborhood of diameter R, measured in the complex Hilbert metric, of $\hat{W}_a$ in $W_{a,\mathbb{C}}$. Let $B_{\vec{\varepsilon}_0}(r)$ denote the complex r-neighborhood of $\vec{\varepsilon}_0$ in $\mathbb{C}^k$.

Following the proof of Theorem 2.1 (especially pages 5254-5255 of [5]), one obtains a lower bound $r > 0$ on the domain of analyticity if there exists $R > 0$ and $0 < \rho < 1$ satisfying the following conditions:

1. For any $a, z \in \mathcal{A}$ and any $\vec{\varepsilon} \in B_{\vec{\varepsilon}_0}(r)$, $f_z^{\vec{\varepsilon}}$ is a contraction, with respect to the complex Hilbert metric, on $\Omega_{a,H}(R)$:

$$\sup_{x \neq y \in \Omega_{a,H}(R)} \left| \frac{d_{z,H}(f_z^{\vec{\varepsilon}}(x), f_z^{\vec{\varepsilon}}(y))}{d_{a,H}(x, y)} \right| \leq \rho < 1.$$

2. for any $\vec{\varepsilon} \in B_{\vec{\varepsilon}_0}(r)$, any $x \in \cup_a \hat{W}_a$ and any $z \in \mathcal{A}$,

$$d_{z,H}(f_z^{\vec{\varepsilon}}(x), f_z^{\vec{\varepsilon}_0}(x)) \leq R(1 - \rho),$$

and

$$d_{z,H}(f_z^{\vec{\varepsilon}}(\pi(\varepsilon)), f_z^{\vec{\varepsilon}_0}(\pi(\varepsilon_0))) \leq R(1 - \rho),$$

(where $\pi(\varepsilon)$ denotes the stationary vector for the Markov chain defined by $\Delta^{\vec{\varepsilon}}$).

3. For any $x \in \Omega_{a,H}(R)$ and $\vec{\varepsilon} \in B_{\vec{\varepsilon}_0}(r)$,

$$\sum_a |r_a^{\vec{\varepsilon}}(x)| \leq 1/\rho.$$

The existence of r, R, ρ follows from Theorem 2.1. In fact, we can choose ρ to be any positive number such that $\max_{a \in \mathcal{A}} \tau(\Delta_a) < \rho < 1$, and small r, R to satisfy condition 1, then smaller r, R, if necessary, to further satisfy conditions 2 and 3.

Let $\Omega_{a,E}(R)$ denote the neighborhood of diameter R, measured in the Euclidean metric, of $\hat{W}_a$ in $W_{a,\mathbb{C}}$. To facilitate the computation, at the expense of obtaining a smaller lower bound, it may be easier to use $\Omega_{a,E}(R)$ instead of $\Omega_{a,H}(R)$; then, the conditions above are replaced with the following conditions:

(1') Condition 1 above with $\Omega_{a,H}(R)$ replaced by $\Omega_{a,E}(R)$ (the map $f_z^{\vec{\varepsilon}}$ is still required to be a contraction under the complex *Hilbert* metric).

(2') Condition 2 above with R on the right hand side of the inequalities replaced by R/K, where $K = \sup_{x \neq y \in \Omega_{a,E}(R), a} \left| \frac{d_{a,E}(x,y)}{d_{a,H}(x,y)} \right|$; note that for R sufficiently small, $0 < K < \infty$ since $d_{a,H}$ and $d_{a,E}$ are equivalent metrics (this in turn follows from the fact that the Euclidean metric and (real) Hilbert metric are equivalent on any compact subset of the interior of the real simplex).

(3') Condition 3 above with $\Omega_{a,H}(R)$ replaced by $\Omega_{a,E}(R)$

2.2 Example for Domain of Analyticity

In the following, we consider hidden Markov processes obtained by passing binary Markov chains through binary symmetric channels with crossover probability ε. Suppose that the Markov chain is defined by a 2×2 stochastic matrix $\Pi = [\pi_{ij}]$. From now through the end of this section, we **assume:**

- $\det(\Pi) > 0$ – and –
- all $\pi_{ij} > 0$ – and –
- $0 < \varepsilon < 1/2$.

We remark that the condition $\det(\Pi) > 0$ is purely for convenience.

Strictly speaking, the underlying Markov process of the resulting hidden Markov process is given by a 4-state matrix (the states are the ordered pairs of a state of Π and a noise state (0 for "noise off" and 1 for "noise on"); see page 5255 of [5]). However, the information contained in each f_a can be reduced to an equivalent map induced by a 2×2 matrix and then reduced to an equivalent function of a single variable as in Example 1.5. We describe this as follows.

Let $a_i = p(z_1^i, y_i = 0)$ and $b_i = p(z_1^i, y_i = 1)$. The pair (a_i, b_i) satisfies the following dynamical system:

$$(a_i, b_i) = (a_{i-1}, b_{i-1}) \begin{bmatrix} p_E(z_i)\pi_{00} & p_E(z_i)\pi_{10} \\ p_E(\bar{z}_i)\pi_{01} & p_E(\bar{z}_i)\pi_{11} \end{bmatrix}.$$

where $p_E(0) = \varepsilon$ and $p_E(1) = 1 - \varepsilon$.

Similar to Example 1.5, let $x_i = a_i/b_i$, we have a dynamical system with just one variable:

$$x_{i+1} = f^{\varepsilon}_{z_{i+1}}(x_i),$$

where

$$f^{\varepsilon}_z(x) = \frac{p_E(z)}{p_E(\bar{z})} \frac{\pi_{00}x + \pi_{10}}{\pi_{01}x + \pi_{11}}, \qquad z = 0, 1$$

starting with

$$x_0 = \pi_{10}/\pi_{01}, \tag{18}$$

which comes from the stationary vector of Π.

It can be shown that

$$p^{\varepsilon}(z_i = 0|z_1^{i-1}) = r^{\varepsilon}_0(x_{i-1}), \qquad p^{\varepsilon}(z_i = 1|z_1^{i-1}) = r^{\varepsilon}_1(x_{i-1}),$$

where

$$r^{\varepsilon}_0(x) = \frac{((1-\varepsilon)\pi_{00} + \varepsilon\pi_{01})x + ((1-\varepsilon)\pi_{10} + \varepsilon\pi_{11})}{x+1}, \tag{19}$$

and

$$r^{\varepsilon}_1(x) = \frac{(\varepsilon\pi_{00} + (1-\varepsilon)\pi_{01})x + (\varepsilon\pi_{10} + (1-\varepsilon)\pi_{11})}{x+1}. \tag{20}$$

Now let $\Omega(R)$ denote the complex R-neighborhood (in Euclidean metric) of the interval

$$S = [S_1, S_2] = \left[\frac{\varepsilon_0\pi_{10}}{(1-\varepsilon_0)\pi_{11}}, \frac{(1-\varepsilon_0)\pi_{00}}{\varepsilon_0\pi_{01}}\right],$$

this interval is the union of $f^{\varepsilon_0}_0([0,\infty])$ and $f^{\varepsilon_0}_1([0,\infty])$; again let $B_{\vec{\varepsilon}_0}(r)$ denote the complex r-neighborhood of a given cross-over probability $\varepsilon_0 > 0$.

The sufficient conditions (1'), (2') and (3') in section 2.1 are guaranteed by the following: there exist $R > 0, r > 0, 0 < \rho < 1$ such that

(1") For any z, $f^{\varepsilon}_z(x)$ is a contraction on $\Omega(R)$ under complex Hilbert metric,

$$\sup_{x \neq y \in \Omega(R)} \left| \frac{\log f^{\varepsilon}_z(x) - \log f^{\varepsilon}_z(y)}{\log x - \log y} \right| \leq \rho < 1.$$

Note that here

$$\log f^{\varepsilon}_z(x) - \log f^{\varepsilon}_z(y) = \log \frac{\pi_{00}x + \pi_{10}}{\pi_{01}x + \pi_{11}} - \log \frac{\pi_{00}y + \pi_{10}}{\pi_{01}y + \pi_{11}}.$$

(2") For any $\varepsilon \in B_{\vec{\varepsilon}_0}(r)$, any $x \in S$ and any z,

$$|\log f^{\varepsilon}_z(x) - \log f^{\varepsilon_0}_z(x)| \leq (R/K)(1-\rho),$$

where

$$K = \sup_{x \neq y \in \Omega(R)} \left| \frac{x - y}{\log x - \log y} \right| = \sup_{x \in \Omega(R)} |x| = S_2 + R.$$

(note that here the second condition in (2') is vacuous since by (18) x_0 does not depend on ε)

(3”) For any $x \in \Omega(R)$ and $\varepsilon \in B_{\vec{\varepsilon}_0}(r)$,

$$|r_0^{\varepsilon}(x)| + |r_1^{\varepsilon}(x)| \leq 1/\rho.$$

By considering extreme cases, the above conditions can be further relaxed to:

(1”’)

$$0 < \frac{\pi_{00}\pi_{11} - \pi_{10}\pi_{01}}{\pi_{01}\pi_{00}(S_1 - R) + \pi_{01}\pi_{10} + \pi_{11}\pi_{00} + \pi_{11}\pi_{10}/(S_2 + R)} \leq \rho.$$

(here we applied the mean value theorem to give an upper bound on $|\log((\pi_{00}x + \pi_{10})/(\pi_{01}x + \pi_{11})) - \log((\pi_{00}y + \pi_{10})/(\pi_{01}y + \pi_{11}))|$)

(2”’)

$$0 < \frac{r}{\varepsilon_0 - r} + \frac{r}{1 - \varepsilon_0 - r} \leq (R/(S_2 + R))(1 - \rho).$$

(here we applied the mean value theorem to give an upper bound on $|\log((1-\varepsilon)/\varepsilon) - \log((1-\varepsilon_0)/\varepsilon_0)|$)

(3”’)

$$0 < \frac{((1-\varepsilon_0+r)\pi_{00} + (\varepsilon_0+r)\pi_{01})(S_2+R) + ((1-\varepsilon_0+r)\pi_{10} + (\varepsilon_0+r)\pi_{11})}{S_1 - R + 1}$$
$$+ \frac{((\varepsilon_0+r)\pi_{00} + (1-\varepsilon_0+r)\pi_{01})(S_2+R) + ((\varepsilon_0+r)\pi_{10} + (1-\varepsilon_0+r)\pi_{11})}{S_1 - R + 1} \leq 1/\rho.$$

In other words, choose r, R and ρ to satisfy the conditions (1”’), (2”’) and (3”’). Then the entropy rate is an analytic function of ε on $|\varepsilon - \varepsilon_0| < r$.

Consider the symmetric case: $\pi_{00} = \pi_{11} = p$ and $\pi_{01} = \pi_{10} = 1-p$. We plot lower bounds on radius of convergence of $H(Z)$ (as a function of ε at $\varepsilon_0 = 0.4$) against p in Figure 1. For a fixed p, the lower bound is obtained by randomly generating many 3-tuples (r, R, ρ) and taking the maximal r from the 3-tuples which satisfy the conditions.

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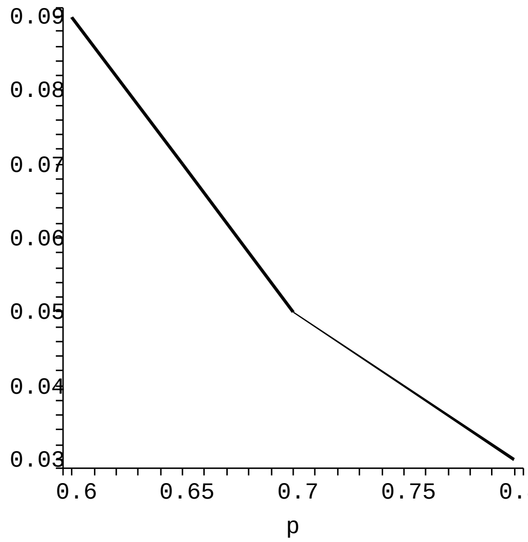

Figure 1: lower bound on radius of convergence as a function of p

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